

Static Cylindrically Symmetric Interior Solutions in $f(R)$ Gravity

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Abstract

We investigate some exact static cylindrically symmetric solutions for a perfect fluid in the metric $f(R)$ theory of gravity. For this purpose, three different families of solutions are explored. We evaluate energy density, pressure, Ricci scalar and functional form of $f(R)$. It is interesting to mention here that two new exact solutions are found from the last approach, one is in particular form and the other is in the general form. The general form gives a complete description of a cylindrical star in $f(R)$ gravity.

Keywords: $f(R)$ gravity; Exact cylindrically symmetric solutions; Perfect fluid.

PACS: 04.50.Kd

1 Introduction

Latest data from CMBR and supernovae surveys show that the energy composition of the universe is as follows: 4% ordinary brayonic matter, 20% dark matter and 76% dark energy [1]-[4]. Dark energy seems to be the most favorite way to explain the accelerated expanding universe as indicated by recent observations. However, the exact nature of dark energy is still unknown. It is assumed to have a large negative pressure. There are some problems in astrophysics and cosmology like the issues of dark energy and dark matter where we experience severe theoretical difficulties.

Many proposals are available in literature [5]-[9] to generalize the theory of general relativity. The $f(R)$ theory of gravity is the simplest generalization such that the Ricci scalar in Einstein-Hilbert action is replaced by an arbitrary function of the Ricci scalar. This generalization leads to complicated field equations as well as of higher order. Due to higher order derivative dependence, we expect to get more exact solutions than general relativity. This theory explains the accelerated expansion of the early universe as well as the current accelerated expansion of the universe.

Recent literature indicates keen interest to investigate exact solutions in $f(R)$ gravity. Multamaki and Vilja [10] discussed static spherically symmetric vacuum solutions. The same authors [11] extended it for perfect fluid and found that pressure and density cannot

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be determined uniquely unless some conditions are imposed. Caramês and de Mello [12] explored static spherically symmetric vacuum solutions in higher dimensions. Capozziello et al. [13] found static spherically symmetric vacuum solutions via Nöether symmetry approach. Hollenstein and Lobo [14] analyzed exact static spherically symmetric solutions by coupling $f(R)$ gravity with nonlinear electrodynamics. Sharif and Shamir [15] explored static plane symmetric vacuum solutions. The same authors [16] found exact solutions for Bianchi types I and V spacetimes in vacuum. Recently, Sharif and Kausar [17] discussed static spherically symmetric dust solutions. The same authors [18] also studied non-vacuum solutions of Bianchi VI_0 universe. In a recent paper, Shojai and Shojai [19] found some static spherically symmetric solutions for perfect fluid satisfying the physical acceptability criteria.

The study of exact solutions in $f(R)$ gravity is mostly restricted to spherical symmetry either vacuum or non-vacuum. One can compare the results with solar system observations based on the Schwarzschild solution. The physical acceptability of these solutions can also be discussed. Azadi et al. [20] investigated static cylindrically symmetric vacuum solutions. Momeni and Gholizade [21] found static cylindrically symmetric vacuum solutions with constant scalar curvature applicable to outside of a string. It would be interesting to extend the study of exact solutions for cylindrically symmetric non-vacuum solutions. Delgaty and Lake [22] gave the physical acceptability criteria for an exact solution in GR as follows:

- isotropy of pressure;
- regularity at the origin;
- positive definiteness of the energy density and pressure at the origin;
- vanishing of pressure at some finite radius;
- the monotonic decrease of the energy density and pressure with increasing radius;
- subluminal sound speed, i.e., $(v_s^2 = \frac{dp}{d\rho} < 1)$.

For $f(R)$ gravity, only the first and last two conditions are accounted.

In this paper, we construct some static cylindrically symmetric interior solutions in $f(R)$ gravity. We consider three types of assumptions depending upon model parameters to explore these solutions in tabular form and discuss them by using the physical acceptability criteria. In particular, we are interested in physically acceptable solutions of a cylindrical star. We discuss the dependence of $f(R)$ functions on the scalar curvature for these solutions. The outline of the paper is as follows: In section 2, we give a brief overview of the metric $f(R)$ gravity and formulate the field equations of cylindrically symmetric spacetime. Section 3 provides solutions of the field equations by using three types of assumptions for particular values of the parameters, given in the form of tables. In the last section, we discuss and summarize the results.

2 Field Equations in $f(R)$ Gravity for Cylindrically Symmetric Spacetime

The modified form of the Einstein-Hilbert action in $f(R)$ gravity is given by [23]

$$S = \int d^4x \sqrt{-g} [f(R) + \kappa \mathcal{L}_m], \quad (1)$$

where g is the trace of the metric tensor $g_{\mu\nu}$, $f(R)$ is a generic function of the Ricci scalar R , κ is the coupling constant in gravitational units and \mathcal{L}_m is the standard matter Lagrangian. Variation of the action with respect to the metric tensor leads to following fourth order partial differential equations

$$F(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu \nabla_\nu F(R) + g_{\mu\nu} \square F(R) = \kappa T_{\mu\nu}, \quad (2)$$

where $T_{\mu\nu}$ is the energy-momentum tensor of matter, $F(R) \equiv df(R)/dR$, $\square \equiv \nabla^\mu \nabla_\mu$ with ∇_μ representing the covariant derivative. Taking trace of the above equation, we obtain

$$F(R)R - 2f(R) + 3\square F(R) = \kappa T. \quad (3)$$

Here R and T are related differentially and not algebraically. This indicates that the field equations of $f(R)$ gravity will admit a larger variety of solutions as compared to GR. Notice that $R = 0$ implies $T = 0$ in GR whereas in $f(R)$ theory, this does not hold. The Ricci scalar curvature function $f(R)$ can be expressed in terms of its derivatives as

$$f(R) = \frac{-\kappa T + F(R)R + 3\square F(R)}{2}. \quad (4)$$

Substituting this value of $f(R)$ in Eq.(2), we obtain

$$R_{\alpha\beta}F(R) - \frac{1}{4}[RF(R) - \square F(R)]g_{\alpha\beta} - \nabla_\alpha \nabla_\beta F(R) = \kappa[T_{\alpha\beta} - \frac{1}{4}(\rho - 3p)g_{\alpha\beta}]. \quad (5)$$

The obtained equation is independent of $f(R)$ and is used to calculate the independent field equations of a system.

The line element of static cylindrically symmetric spacetime is given by [24]

$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2(d\theta^2 + \alpha^2 dz^2), \quad (6)$$

where A , B are the metric coefficients and α has dimensions of $1/r$. The energy-momentum tensor for perfect fluid is

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu}, \quad (7)$$

here $u_\mu = \delta_\mu^0$ is the four-velocity in co-moving coordinates, ρ is the density and p is the pressure of the fluid. The corresponding Ricci scalar has the form

$$R = \frac{1}{X} \left[A'' + \frac{4A'}{r} - \frac{X'}{X} \left(\frac{A'}{2} + \frac{2A}{r} \right) + \frac{2A}{r^2} \right], \quad (8)$$

where $X = AB$ and prime denotes derivative with respect to the radial coordinate r . Using Eq.(5), we get the following two independent equations

$$-2rF'' + r\frac{X'}{X}F' + 2F\frac{X'}{X} - \frac{2r\kappa X}{A}(\rho + p) = 0, \quad (9)$$

$$A'' + \left(\frac{F'}{F} - \frac{X'}{X}\right)\left(A' - \frac{2A}{r}\right) - \frac{2A}{r^2} - \frac{2\kappa X}{F}(\rho + p) = 0. \quad (10)$$

The conservation equation, $T_{\mu;\nu}^\nu = 0$, leads to

$$\frac{A'}{A} = -\frac{2p'}{\rho + p}. \quad (11)$$

Thus we have a system of three non-linear differential equations with five unknown functions, namely, $F(r)$, $\rho(r)$, $p(r)$, $A(r)$ and $B(r)$.

3 Solutions of the Field Equations

This section is devoted to discuss the solutions of the field equations. The Ricci scalar depends on $A(r)$ and $B(r)$, so it would be much difficult to solve the field equations involving $f(R)$. Consequently, analytical solutions of Eqs.(9)-(11) could not be found in closed form. Here, we use Tolman's method [25] which constructs a variety of explicit solutions of the Einstein field equations. Tolman introduced a relation for the metric components depending on radial coordinates to solve the set of equations for perfect fluid. This procedure yields pressure and density such that they may or may not be physically acceptable. We explore cylindrically symmetric exact solutions in terms of energy density, pressure and functions of R . In order to solve the field equations, we use three types of assumptions [10, 19] for X and F given below:

- $X = X_0$, $F = F_0 r^n$,
- $X = X_0 r^m$, $F = F_0$,
- $X = X_0 r^m$, $F = F_0 r^n$,

where X_0 , F_0 , n and m are arbitrary constants. For any choice of m and n , a number of interesting solutions for A , ρ , p and R are obtained along with the corresponding function $f(R)$.

3.1 Type I: $X = X_0$, $F = F_0 r^n$

Inserting $X = X_0$, $F = F_0 r^n$ in Eqs.(9) and (10) and combining the resulting equations, we obtain

$$A'' + \frac{n}{r}A' + \frac{2n^2 - 4n - 2}{r^2}A = 0. \quad (12)$$

Its general solution is

$$A = A_0 r^m, \quad m = \frac{1}{2} \left(1 - n \pm \sqrt{9 + 14n - 7n^2}\right), \quad (13)$$

where A_0 is a constant. Using this value in Eqs.(9)-(11), it follows that

$$p = p_0 + p_1 r^{m+n-2}, \quad p_1 = \frac{F_0 A_0 m n (n-1)}{2\kappa X_0 (n+m-2)}, \quad (14)$$

where p_0 is another constant. Further

$$\rho = -p_0 + \rho_1 r^{m+n-2}, \quad \rho_1 = -\frac{F_0 A_0 n (n-1) (2n+3m-4)}{2\kappa X_0 (n+m-2)}. \quad (15)$$

The Ricci scalar takes the form

$$R = R_1 r^{m-2}, \quad R_1 = \frac{A_0}{X_0} (m^2 + 3m + 2). \quad (16)$$

We can write some exact solutions for particular values of m and n given in **Table 1**. The last two columns have functional form of $f(R)$ and speed of sound, respectively. We have already assumed the isotropy of pressure. This type does not provide any solution which satisfies the physically acceptable criteria completely as the solutions I, III, V, VI and VII indicate squared sound velocity as negative. The solutions II and IV do not have monotonic decrease in energy density and pressure with increasing radius.

Table 1: Type I solutions with $X = X_0$, $F = F_0 r^n$.

No.	m	n	$A(r)$	$p(r)$	$\rho(r)$	$R(r)$	$f(R)$	$v_s^2 = \frac{dp}{d\rho}$
I	-3	1	$A_0 r$	$p_0 + \frac{A_0 F_0 r}{\kappa X_0}$	$-p_0 - \frac{3A_0 F_0 r}{\kappa X_0}$	$\frac{6A_0}{X_0 r}$	$f_0 - \frac{5F_0 R}{12}$	$-\frac{1}{3}$
II	-2	0	$A_0 r^2$	p_0	$-p_0$	$\frac{12A_0}{X_0}$	any regular	< 1
III	-2	1	$\frac{A_0}{r^2}$	p_0	$-p_0$	0	any regular	-1
IV	0	1	$A_0 r$	p_0	$-p_0$	$\frac{6A_0}{X_0 r}$	any regular	< 1
V	0	2	$A_0 r^2$	p_0	$-p_0$	$\frac{12A_0}{X_0}$	any regular	-1
VI	2.5	1	$A_0 r$	$p_0 + \frac{5A_0 F_0 r^{3/2}}{4\kappa X_0}$	$-p_0 + \frac{15A_0 F_0 r^{3/2}}{2\kappa X_0}$	$\frac{6A_0}{X_0 r^3}$	$f_0 - \frac{F_0 (3R^{-1/3} - 5R)^{1/2}}{6}$	$-\frac{1}{4}$
VII	2.5	2	$\frac{A_0}{r}$	$p_0 - \frac{15A_0 F_0}{4\kappa X_0 \sqrt{r}}$	$-p_0 + \frac{15A_0 F_0}{4\kappa X_0 \sqrt{r}}$	0	any regular	$-\frac{1}{2}$

3.2 Type II: $X = X_0 r^m$, $F = F_0$

Now replacing these values in Eqs.(9) and (10), it follows that

$$A'' - \frac{m}{2r} A' - \frac{2+m}{r^2} A = 0, \quad (17)$$

which leads to the following solution

$$A = A_0 r^n, \quad n = \frac{1}{4} \left(2 + m \pm \sqrt{m^2 + 20m + 36} \right). \quad (18)$$

Using this solution, we can write pressure, density and Ricci scalar as

$$p = p_0 + p_1 r^{n-m-2}, \quad p_1 = -\frac{F_0 A_0 m n}{2\kappa X_0 (n-m-2)}, \quad (19)$$

$$\rho = -p_0 + \rho_1 r^{n-m-2}, \quad \rho_1 = \frac{F_0 A_0 m (3n-2m-4)}{2\kappa X_0 (n-m-2)}, \quad (20)$$

$$R = R_1 r^{n-m-2}, \quad R_1 = \frac{A_0}{X_0} (n^2 + 3n - 2m - nm + 2). \quad (21)$$

For the same choices of m and n taken for type I solutions, we construct a **Table 2** for this type of solutions given below. Following the physical acceptability criteria, we see that solutions II and V give negative squared sound velocity, while it becomes 1 for solution IV. Solutions I and III do not show decrement in energy density and pressure by increasing the value of radius. Only solution VI satisfies complete criteria but the Ricci scalar turns out to be radial dependent. Hence, we could not obtain any acceptable solution for the constant curvature condition.

Table 2: Type II solutions with $X = X_0 r^m$, $F = F_0$.

No.	m	n	$A(r)$	$p(r)$	$\rho(r)$	$R(r)$	$f(R)$	$v_s^2 = \frac{dp}{d\rho}$
I	-2	0	A_0	p_0	$-p_0$	$\frac{6A_0}{X_0}$	$f_0 - \frac{F_0 R}{2}$	< 1
II	-2	1	$A_0 r$	$p_0 + \frac{A_0 F_0 r}{\kappa X_0}$	$-p_0 - \frac{3A_0 F_0 r}{\kappa X_0}$	$\frac{12A_0 r}{X_0}$	$f_0 + \frac{5F_0 R}{12}$	$-\frac{1}{3}$
III	-3	1	$A_0 r$	$p_0 + \frac{3A_0 F_0 r^2}{4\kappa X_0}$	$-p_0 - \frac{15A_0 F_0 r^2}{4\kappa X_0}$	$\frac{15A_0 r^2}{X_0}$	$f_0 - \frac{3F_0 R}{10}$	< 1
IV	0	1	$A_0 r$	p_0	$-p_0$	$\frac{6A_0}{X_0 r}$	any regular	1
V	0	2	$A_0 r^2$	p_0	$-p_0$	$\frac{12A_0}{X_0}$	any regular	-1
VI	2.5	2	$A_0 r^2$	$p_0 + \frac{A_0 F_0}{4\kappa X_0 r^{5/2}}$	$-p_0 + \frac{6A_0 F_0}{5\kappa X_0 r^{5/2}}$	$\frac{2A_0}{X_0 r^{5/2}}$	$f_0 - \frac{23F_0 R}{20}$	< 1

3.3 Type III: $X = X_0 r^m$, $F = F_0 r^n$

This type of assumption deals with radial dependent expressions both for X and F , i.e.,

$$X = X_0 r^m \quad F = F_0 r^n$$

Replacing these values in Eqs.(9) and (10), it follows that

$$A'' - \frac{2n-m}{2r}A' - \frac{2n^2-4n-m-2-nm}{r^2}A = 0, \quad (22)$$

which has the following general solution

$$A = A_1 r^{l_1} + A_2 r^{l_2}, \quad (23)$$

where A_1 , A_2 are constants and l_1 , l_2 are given as

$$l_1 = \frac{1}{4} \left(m - 2n + \sqrt{32 + 16m + m^2 + 64n + 12mn - 28n^2} \right),$$

$$l_2 = \frac{1}{4} \left(m - 2n - \sqrt{32 + 16m + m^2 + 64n + 12mn - 28n^2} \right).$$

Inserting the above values in Eqs.(9)-(11), the corresponding p , ρ and R are obtained as

$$p = p_0 + p_1 r^{l_1+n-m-2} + p_2 r^{l_2+n-m-2} \quad (24)$$

with

$$p_1 = \frac{F_0 A_1 l_1 (2n^2 - 2n - 2m - mn)}{4\kappa X_0 (l_1 + n - m - 2)}, \quad p_2 = \frac{F_0 A_2 l_2 (2n^2 - 2n - 2m - mn)}{4\kappa X_0 (l_2 + n - m - 2)}.$$

Also,

$$\rho = -p_0 + \rho_1 r^{l_1+n-m-2} + \rho_2 r^{l_2+n-m-2}, \quad (25)$$

where

$$\begin{aligned} \rho_1 &= -\frac{F_0 A_1 (3l_1 + 2n - 2m - 4)(2n^2 - 2n - 2m - mn)}{4\kappa X_0 (l_1 + n - m - 2)}, \\ \rho_2 &= -\frac{F_0 A_2 (3l_2 + 2n - 2m - 4)(2n^2 - 2n - 2m - mn)}{4\kappa X_0 (l_2 + n - m - 2)}. \end{aligned}$$

The corresponding Ricci scalar is

$$R = R_1 r^{l_1-m-2} + R_2 r^{l_2-m-2}, \quad (26)$$

where

$$R_1 = \frac{A_1}{2X_0} (2l_1^2 + (6-m)l_1 + 4m), \quad R_2 = \frac{A_2}{2X_0} (2l_2^2 + (6-m)l_2 + 4m).$$

We formulate solutions corresponding to l_1 and l_2 in two different tables. Firstly, we calculate solutions for l_1 given in **Table 3**.

Table 3: Type III solutions with $X = X_0 r^m$, $F = F_0 r^n$ corresponding to l_1 .

No.	m	n	$A(r)$	$p(r)$	$\rho(r)$	$R(r)$	$f(R)$	$v_s^2 = \frac{dp}{d\rho}$
I	-12	-1	$\frac{A_1}{r}$	$p_0 - \frac{A_1 F_0 r^8}{2\kappa X_0}$	$-p_0 - \frac{15A_1 F_0 r^8}{2\kappa X_0}$	$\frac{-32A_1 r^9}{X_0}$	$f_0 - aF_0 R^{8/9}$ where $a = \frac{1}{2^{5/9}} (\frac{A_1}{X_0})^{1/9}$	$-\frac{1}{8}$
II	-10	-1	$\frac{A_1}{r^2}$	$p_0 - \frac{7A_1 F_0 r^5}{5\kappa X_0}$	$-p_0 - \frac{28A_1 F_0 r^5}{5\kappa X_0}$	$\frac{-32A_1 r^6}{X_0}$	$f_0 - abF_0 R^{5/6}$ where $a = \frac{117}{5(2)^{25/6}},$ $b = (\frac{A_1}{X_0})^{1/6}$	$-\frac{1}{4}$
III	-2	0	A_1	p_0	$-p_0$	$\frac{-4A_1}{X_0}$	any	< 1
IV	1	3	A_1	p_0	$-p_0$	$\frac{2A_1}{X_0 r^3}$	any regular	< 1
V	4	1	$A_1 r^4$	$p_0 + \frac{12A_1 F_0 r^{-1}}{\kappa X_0}$	$-p_0 - \frac{6A_1 F_0 r^{-1}}{\kappa X_0}$	$\frac{28A_1}{X_0 r^2}$	$f_0 - \frac{17}{4\sqrt{7}} a F_0 \sqrt{R}$ where $a = \sqrt{\frac{A_1}{X_0}}$	2
VI	6	2	$A_1 r^5$	$p_0 + \frac{25A_1 F_0 r^{-1}}{\kappa X_0}$	$-p_0 - \frac{15A_1 F_0 r^{-1}}{\kappa X_0}$	$\frac{37A_1}{X_0 r^3}$	$f_0 + abF_0 \sqrt[3]{R}$ where $a = \frac{63}{2\sqrt[3]{37}},$ $b = (\frac{A_1}{X_0})^{2/3}$	< 1
VII	8	5	$A_1 r^4$	$p_0 + \frac{16A_1 F_0 r^{-1}}{\kappa X_0}$	$-p_0 - \frac{8A_1 F_0 r^{-1}}{\kappa X_0}$	$\frac{26A_1}{X_0 r^6}$	$f_0 + abF_0 R^{1/6}$ where $a = \frac{51}{\sqrt[5]{2}\sqrt[5]{13}},$ $b = (\frac{A_1}{X_0})^{5/6}$	2

Since we have already assumed isotropic pressure, so this condition automatically holds for all the solutions. We see that solutions III and IV have subluminal speed but energy density and pressure do not decrease with the increase in radius. The solutions I and II show

decrease in density and pressure as radius increases but these give negative squared sound velocity. For solutions V and VII, the energy density and isotropic pressure decrease when the radius increases and $v_s^2 > 1$. The solution VI fulfills the criteria of physical acceptability properly. Hence this is the only acceptable solution obtained by using this assumption.

Table 4: Type III solutions with $X = X_0 r^m$, $F = F_0 r^n$ corresponding to l_2 .

$No.$	m	n	$A(r)$	$p(r)$	$\rho(r)$	$R(r)$	$f(R)$	$v_s^2 = \frac{dp}{d\rho}$
<i>I</i>	-2	0	$\frac{A_2}{r}$	$p_0 + \frac{A_2 F_0 r^{-1}}{\kappa X_0}$	$-p_0 - \frac{3A_2 F_0 r^{-1}}{\kappa X_0}$	$\frac{-7A_2}{X_0 r}$	$f_0 - \frac{17}{14}R$	< 1
<i>II</i>	4	1	$\frac{A_2}{r^3}$	$p_0 - \frac{9A_2 F_0 r^{-8}}{8\kappa X_0}$	$-p_0 + \frac{57A_2 F_0 r^{-8}}{8\kappa X_0}$	$\frac{14A_2}{X_0 r^9}$	$f_0 - abF_0 R^{(8/9)}$ where $a = \frac{91}{2^{26/9} \times 7^{8/9}}$, $b = (\frac{A_2}{X_0})^{1/9}$	< 1
<i>III</i>	4	3	$\frac{A_2}{r^4}$	$p_0 - \frac{8A_2 F_0 r^{-7}}{7\kappa X_0}$	$-p_0 + \frac{36A_2 F_0 r^{-7}}{7\kappa X_0}$	$\frac{-10A_2}{X_0 r^{10}}$	$f_0 - abF_0 R^{7/10}$ where $a = \frac{-205 \times 2^{3/10}}{7 \times (5)^{7/10}}$, $b = (\frac{A_2}{X_0})^{3/10}$	< 1
<i>IV</i>	6	2	$\frac{A_2}{r^4}$	$p_0 - \frac{2A_2 F_0 r^{-10}}{\kappa X_0}$	$-p_0 + \frac{12A_2 F_0 r^{-10}}{\kappa X_0}$	$\frac{28A_2}{X_0 r^{12}}$	$f_0 - abF_0 R^{5/6}$ where $a = \frac{3 \times 2^{7/3}}{7^{5/6}}$, $b = (\frac{A_2}{X_0})^{1/6}$	< 1
<i>V</i>	6	5	$\frac{A_2}{r^4}$	$p_0 - \frac{2A_2 F_0 r^{-7}}{7\kappa X_0}$	$-p_0 + \frac{9A_2 F_0 r^{-7}}{7\kappa X_0}$	$\frac{28A_2}{X_0 r^{12}}$	$f_0 - abF_0 R^{7/12}$ where $a = \frac{165}{2^{1/6} \times 7^{19/12}}$, $b = (\frac{A_2}{X_0})^{5/12}$	< 1
<i>VI</i>	8	5	$\frac{A_2}{r^5}$	$p_0 - \frac{2A_2 F_0 r^{-10}}{\kappa X_0}$	$-p_0 + \frac{10A_2 F_0 r^{-10}}{\kappa X_0}$	$\frac{46A_2}{X_0 r^{15}}$	$f_0 + abF_0 \sqrt[3]{R^2}$ where $a = \frac{123}{2^{5/3} \times (23)^{2/3}}$, $b = (\frac{A_2}{X_0})^{1/3}$	< 1

The solutions corresponding to l_2 are shown in **Table 4**. In this table, the increment of radius monotonically decreases pressure and density. Moreover, sound speed is subluminal. Thus we can conclude that the most acceptable solution of a cylindrical star in $f(R)$ gravity

is generally described as:

$$\begin{aligned}
A(r) &= A_2 r^l, \\
p(r) &= p_0 + p_2 r^{l+n-m-2}, \\
\rho(r) &= -p_0 + \rho_2 r^{l+n-m-2}, \\
R(r) &= R_2 r^{l-m-2}, \quad R_2 = \frac{A_2}{2X_0}(2l^2 + (6-m)l + 4m), \\
f(R) &= f_0 + abF_0 R^{1+k}, \quad k = \frac{n}{l-m-2}. \\
a &= \frac{1}{2} \left\{ (2l^2 + (6-m)l + 4m) - \frac{3n}{2}(l-2m-4) - 2n(n-1) \right. \\
&\quad \left. - \frac{(10l+6n-6m-4)(2n^2-2n-2m-mn)}{l+n-m-2} \right\} \{2l^2 + (6-m)l + 4m\}^{-(1+k)}, \\
b &= \left(\frac{A_2}{X_0} \right)^{-k}, \quad l = \frac{1}{4} \left(m-2n - \sqrt{32 + 16m + m^2 + 64n + 12mn - 28n^2} \right).
\end{aligned}$$

4 Summary

This paper is devoted to study the static cylindrically symmetric solutions in metric $f(R)$ gravity for perfect fluid. In this theory, the field equations are highly non-linear and complicated which cannot be handled analytically without taking some assumptions. For this purpose, we confine ourselves to only those solutions which are closed and are found analytically without use of any numerical data, perturbation or approximation method. The corresponding form of $f(R)$ is reconstructed which does not depend algebraically on $R(r)$ and $f(r)$. Physical acceptability criteria is applied to check the physically acceptable solutions. We can summarize the results as follows:

- First type of assumption provides some solutions given in **Table 1** but none of them satisfies physical acceptability measures. Thus no new solution is obtained.
- Second type is based on constant scalar curvature condition which gives some new solutions in **Table 2**. It is noted that some of them are physically acceptable but do not fulfill the imposed conditions.
- For the third family of solutions, we formulate two tables corresponding to the parameters l_1 and l_2 . We obtain one solution in **Table 3** which satisfies the physically acceptable criterion, whereas all the solutions in **Table 4** fulfill the conditions for a physically acceptable solution. The later gives the static cylindrically symmetric interior solutions in the general form describing properties of a cylindrical star for $f(R)$ gravity.

Since the exact solutions of $f(R)$ cosmological models are used to explore dark energy and dust matter phases. Thus, these solutions may provide a pointer towards the unknown nature of dark energy and dark matter. These solutions might be useful at some stage

to overcome the theoretical difficulties in the context of cosmology and astrophysics. The applicability of the solutions could be tested by comparing with cosmology and some other constraints. It would be interesting to investigate solutions for non-static spacetimes and also with different types of fluid.

References

- [1] Riess, A.G. et al.: *Astron. J.* **116**(1998)1009.
- [2] Eisenstein, D.J. et al.: *Astrophys. J.* **633**(2005)560.
- [3] Astier, P. et al.: *Astron. Astrophys.* **447**(2006)31.
- [4] Spergel, D.N. et al.: *Astrophys. J. Suppl. Ser.* **170**(2007)377.
- [5] Kiess, T.E.: *Class. Quantum Grav.* **26**(2009)025011.
- [6] Nojiri, S. and Odintsov, S.D.: *Phys. Rep.* **505**(2011)59.
- [7] Nojiri, S. and Odintsov, S.D.: *Int. J. Geom. Meth. Mod. Phys.* **4**(2007)115.
- [8] Clifton, T. et al.: *Phys. Rep.* 513(2012)1.
- [9] Soitiriou, T.P.: *Rev. Mod. Phys.* **82**(2010)451.
- [10] Multamäki, T. and Vilja, I.: *Phys. Rev.* **D74**(2006)064022.
- [11] Multamäki, T. and Vilja, I.: *Phys. Rev.* **D76**(2007)064021.
- [12] Caramês, T.R.P. and de Mello, B.E.R.: *Eur. Phys. J.* **C64**(2009)113.
- [13] Capozziello, S., Stabile, A. and Troisi, A.: *Class. Quantum Grav.* **24**(2007)2153.
- [14] Hollenstein, L. and Lobo, F.S.N.: *Phys. Rev.* **D78**(2008)124007.
- [15] Sharif, M. and Shamir, M.F.: *Mod. Phys. Lett.* **A25**(2010)1281.
- [16] Sharif, M. and Shamir, M.F.: *Class. Quantum Grav.* **26**(2009)235020.
- [17] Sharif, M. and Kausar, H.R.: *J. Phys. Soc. Jpn.* **80**(2011)044004.
- [18] Sharif, M. and Kausar, H.R.: *Astrophys. Space Sci.* **332**(2011)463.
- [19] Shojai, A. and Shojai, F.: *Gen. Relativ. Gravit.* **44**(2012)211.
- [20] Azadi, A., Momeni, D. and Nouri-Zonoz, M.: *Phys. Lett.* **B670**(2008)210.
- [21] Momeni, D. and Gholizade, H.: *Int. J. Mod. Phys.* **D18**(2009)1719.
- [22] Delgaty, M.S.R. and Lake, K.: *Comput. Phys. Commun.* **115**(1998)381.
- [23] Capozziello, S., Cardone, V.F. and Troisi, A.: *Phys. Rev.* **D71**(2005)043503.
- [24] Chao-Guang, H.: *Acta Phys. Sin.* **4**(1995)617.
- [25] Tolman, R.C.: *Phys. Rev.* **55**(1939)364.